IMO Algebra Questions

Level: Intermediate   Ref No: M02   Puzz Points: 10

[Cayley 2004 Q2] Mars, his wife Venus and grandson Pluto have a combined age of 192. The ages of Mars and Pluto together total 30 years more than Venus' age. The ages of Venus and Pluto together total 4 years more than Mars's age. Find their three ages.

Solution: $\boldsymbol{m = 94, v = 81, p = 17}$

Level: Intermediate   Ref No: M09   Puzz Points: 15

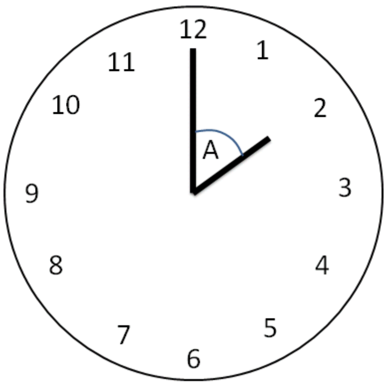
[Hamilton 2004 Q5] A $s \times s$ square, where s is an odd integer, is divided into unit squares (1×1). All the unit squares along the edges and the two diagonals of the $s \times s$ square are discarded. Find a fully simplified expression, in terms of s, for the number of unit squares remaining.

Solution: $\boldsymbol{(s-3)^2}$ **or** $\boldsymbol{s^2 - 6s + 9}$

Level: Intermediate   Ref No: M11   Puzz Points: 18

[Hamilton 2004 Q5]

(a) What is the angle A between the hands of the clock at 2 o'clock?
(b) What is the next time after this that the angle between the hands is equal to A?



Solution: (a) $\boldsymbol{60°}$ **(b) 21** $\boldsymbol{\frac{9}{11}}$ **minutes after 2 o'clock**

Level: Intermediate   Ref No: M14   Puzz Points: 20

[Maclaurin 2004 Q2] An *arithmetic* sequence is one in which the difference between successive terms remains constant (for example, 4, 7, 10, 13, …). Suppose that a right-angled triangle has the property that the lengths of its sides form an arithmetic sequence. Prove that the sides of the triangle are in the ratio 3:4:5.

Solution: If sides are $a-d, a, a+d$, then by Pythagoras, we eventually find the three sides of the triangles are $3d, 4d, 5d$, and thus in ratio $3:4:5$.

Level: Intermediate   Ref No: M16   Puzz Points: 20

[Maclaurin 2004 Q4] Solve the simultaneous equations:

$$x+y=3$$
$$x^3+y^3=9$$

Solution: $(x,y)=(1,2) or\ (2,1)$

Level: Intermediate   Ref No: M23   Puzz Points: 13

[Maclaurin 2004 Q5] The magician Mij has 140 green balls and 140 red balls. To perform a trick, Mij places all the balls in two bags. In the black bag there are twice as many green balls as red balls. In the black bag there are twice as many green balls as red balls. In the white bag the number of red balls is a multiple of the number of green balls. Neither bag is empty. Determine all the ways in which Mij can place the balls in the two bags in order to perform the trick.

Solution: (black bag red, black bag green, white bag red, white bag green) $=$
$(56, 112, 84, 28), (60, 120, 80, 20), (68, 136, 72, 4)$

Level: Intermediate   Ref No: M24   Puzz Points: 13

[Cayley 2006 Q6] A mathematician has a full one-litre bottle of concentrated orange squash, a large container and a tap. He first pours half of the bottle of orange squash into the container. Then he fills the bottle from the tap, shakes well, and pours half of the resulting mixture into the container. He then repeats this step over and over again: filling the bottle from the tap each time, shaking the mixture well, and then pouring half of the contents into the container.

Suppose that on the final occasion he fills the bottle from the tap and empties it completely into the container. How many times has he filled the bottle from the tap if the final mixture consists of 10% orange squash concentrate?

Solution: 18

Level: Intermediate   Ref No: M27   Puzz Points: 15

[Hamilton 2006 Q3] James, Alison and Vivek go into a shop to buy some sweets. James spends £1 on four Fudge Bars, a Sparkle and a Chomper. Alison spends 70p on three Chompers, two Fudge Bars and a Sparkle. Vivek spends 50p on two Sparkles and a Fudge Bar.
What is the cost of a Sparkle?

Solution: 15p (Fudge bar costs 20p and Chomper costs 5p)

Level: Intermediate   Ref No: M29   Puzz Points: 18

[Hamilton 2006 Q5] The Principal of Abertawe Academy plans to employ more teachers. If she employs 10 new teachers, then the number of pupils per teacher will be reduced by 5. However, if she employs 20 new teachers, then the number of pupils per teacher will be reduced by 8.

How many pupils are there at Abertawe Academy?

Solution: 600

Level: Intermediate   Ref No: M32   Puzz Points: 20

[Maclaurin 2006 Q2] Find all integer values that satisfy the following equations:

$$x^2 + y^2 = x - 2xy + y$$
$$x^2 - y^2 = x + 2xy - y$$

Solution: (0, 0), (0, 1), (1,0), (1, -1)

Level: Intermediate   Ref No: M38   Puzz Points: 10

[Cayley 2007 Q2] Before the last of a series of tests, Sam calculated that a mark of 17 would enable her to average 80 over the series, but that a mark of 92 would raise her average mark over the series to 85. How many tests were there in the series?

Solution: 15 tests

Level: Intermediate   Ref No: M54   Puzz Points: 23

[Maclaurin 2007 Q6] Find all real values of *x* and *y* that satisfy the equations:

$$x^4 - y^4 = 5$$
$$x + y = 1$$

Solution: $\left(\frac{3}{2}, -\frac{1}{2}\right)$

Level: Intermediate Ref No: M57 Puzz Points: 10

[Cayley 2011 Q3] At dinner on a camping expedition, each tin of soup was shared between 2 campers, each tin of meatballs was shared between 3 campers and each tin of chocolate pudding was shared between 4 campers. Each camper had all three courses and all tins were emptied. The camp leader opened 156 tins in total. How many campers were on the expedition?

Solution: 144 campers

Level: Intermediate Ref No: M61 Puzz Points: 15

[Hamilton 2011 Q1] If Julie gave £12 to her brother Garron, then he would have half the amount that she would have. If instead Garron gave £12 to his sister Julie, then she would have three times the amount that he would have.

How much money do they each have?

Solution: Julie has £204, Garron has £84

Level: Intermediate Ref No: M83 Puzz Points: 18

[Hamilton 2008 Q6] Find all solutions to the simultaneous equations:

$$x^2 - y^2 = -5$$
$$2x^2 + xy - y^2 = 5$$

Solution: $\boldsymbol{x = 2, y = 3}$ **and** $\boldsymbol{x = -2. y = -3}$

Level: Intermediate Ref No: M90 Puzz Points: 10

[Cayley 2009 Q1] An aquarium contains 280 tropical fish of various kinds. If 60 more clownfish were added to the aquarium, the proportion of clownfish would be doubled.

How many clownfish are in the aquarium?

Solution: 42 clownfish

Level: Intermediate Ref No: M92 Puzz Points: 10

[Cayley 2009 Q3] Two different rectangles are placed together, edge-to-edge, to form a large rectangle. The length of the perimeter of the large rectangle is $\frac{2}{3}$ of the total perimeter of the original two rectangles.

Prove that the final rectangle is in fact a square.

Level: Intermediate Ref No: M98 Puzz Points: 15

[Hamilton 2009 Q4] Four positive integers a, b, c, and d are such that:

The sum of a and b is half the sum of c and d.
The sum of a and c is twice the sum of b and d.
The sum of a and d is one and a half times the sum of b and c.
What is the smallest possible values of $a + b + c + d$?

Solution: 30

Level: Intermediate Ref No: M100 Puzz Points: 18

[Hamilton 2009 Q6] Two different cuboids are placed together, face-to-face, to form a large cuboid. The surface area of the large cuboid is $\frac{3}{4}$ of the total surface area of the original two cuboids.

Prove that the lengths of the edges of the large cuboid may be labelled x, y and z, where

$$\frac{2}{z} = \frac{1}{x} + \frac{1}{y}$$

Level: Intermediate Ref No: M101 Puzz Points: 20

[Maclaurin 2009 Q1] Five numbers are arranged in increasing order. As they get larger the difference between adjacent numbers doubles.

The average of the five numbers is 11 more than the middle number. The sum of the second and fourth numbers is equal to the largest number.

What is the largest number?

Solution: 110

Level: Intermediate   Ref No: M103   Puzz Points: 20

[Maclaurin 2009 Q3] Solve the simultaneous equations

$$\frac{5xy}{x+y} = 6$$

$$\frac{5xz}{x+z} = 3$$

$$\frac{3yz}{y+z} = 2$$

Solution: $\boldsymbol{x = 3, y = 2, z = 1}$

Level: Intermediate   Ref No: M118   Puzz Points: 20

[Maclaurin 2010 Q1] How many different ways are there to express $\frac{2}{15}$ in the form $\frac{1}{a} + \frac{1}{b}$, where a and b are positive integers with $a \leq b$?

Solution: 5

Level: Intermediate   Ref No: M120   Puzz Points: 20

[Maclaurin 2010 Q3] Solve the equations:

$$x + xy + x^2 = 9$$
$$y + xy + y^2 = -3$$

Solution: $\boldsymbol{x = 3, y = -1}$ and $\boldsymbol{x = -\frac{9}{2}, y = \frac{3}{2}}$

Level: Intermediate   Ref No: M135   Puzz Points: 18

[Hamilton 2005 Q6] Four friends, Anna, Bob, Claire and Duncan, all have different heights and the sum of their heights if 6m 72cm. Anna is 8cm taller than Claire, and Bob is 4cm shorter than Duncan. The sum of the heights of the tallest and shortest of the friends is 2cm more than the sum of the heights of the other two. Find the height of each person.

Solution: Anna $\boldsymbol{1m}$ 72$\frac{1}{2}$$\boldsymbol{cm}$, Bob 1m 65$\frac{1}{2}$ cm, Claire 1m 64 $\frac{1}{2}$cm, Duncan 1m 69$\frac{1}{2}$ cm

Level: Intermediate   Ref No: M137   Puzz Points: 20

[Maclaurin 2005 Q2] Solve the simultaneous equations:

$$p + pr + pr^2 = 28$$
$$p^2r + p^2r^2 + p^2r^3 = 224$$

Solution: $(\boldsymbol{p}, \boldsymbol{r}) = (\mathbf{4}, \mathbf{2})$ **or** $\left(\mathbf{16}, \frac{\mathbf{1}}{\mathbf{2}}\right)$